A stationary-phase approximation to the ship-wave pattern

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Scorer (1950) has described an extension of the method of stationary phase. This method is used to derive an approximation to the wave pattern at large distances from the ship in a simple manner. The approximation is valid in the vicinity of the critical angle and the nature of the disturbance in this region is readily seen. The divergent and transverse wave systems in a critical region are shown, together with their variation of amplitude and phase.

1. Introduction

The pattern of gravity waves set up behind a ship on a steady course has been discussed by many writers. Hogner (1923) has given a first approximation to this pattern, and recently Ursell (1960) has given an asymptotic series solution for the critical region for a point disturbance. In this paper a first approximation is derived in a simpler manner.

An integral expression for the wave motion caused by a ship in deep water has been given, for example, by Hogner (1923, equation (51)). The form which we will use here is

$$\zeta = \operatorname{Im} \int_{-\rho}^{\infty} \psi(k)\, e^{ix\phi(k)}\, dk,$$

where $\phi(k) = \cosh k + \frac{1}{2}\tan\theta \sinh 2k$, and where $\psi(k)$ is some function which depends on the shape of the ship. For the present purpose we may assume that ρ is infinite. ζ is the surface displacement at a point with Cartesian co-ordinates (x, y) or polar co-ordinates (r, θ), where the x-axis ($\theta = 0$) is drawn on the surface against the direction of motion. The unit of distance is U^2/g, where U is the steady velocity of the ship. Terms of order r^{-1} have been omitted in this representation.

The usual method of stationary phase gives a first approximation of ζ in the region $|\theta| < \tan^{-1}(1/2\sqrt{2}) = \theta_c$, but it fails in the vicinity of $\theta = \theta_c$ since $\phi'(k)$ and $\phi''(k)$ vanish together along this line. (The dashes denote differentiation with respect to k.) Scorer's method overcomes this difficulty by replacing $\phi(k)$ with parabolas of the third order applied at the stationary values of ϕ. This leads to an approximation in terms of the Airy integral Ai(z) and the conjugate integral Gi(z):

$$\operatorname{Ai}(z) + i\operatorname{Gi}(z) = \frac{1}{\pi}\int_0^{\infty} \exp\{i(zu + \tfrac{1}{3}u^3)\}\, du.$$

Ai(z) has been tabulated by Miller (1946). Scorer (1950) tabulates Gi(z) for $|z| \leqslant 10$, and for larger values there is an asymptotic expression valid to the accuracy of the tables.

2. The approximation

For this purpose the range of integration is suitably divided at k_I, where $\phi''(k_I) = 0$. The function $\phi(k)$ has two turning points when $|\theta| < \theta_c$ and these are denoted by the suffixes T and D, where $k_D < k_I < k_T$. The approximation to $\phi(k)$ is then as follows:

$$\text{(i)}\quad \phi(k) \doteqdot \phi_T + \tfrac{1}{2}\phi''_T(k-k_T)^2 + \tfrac{1}{6}\phi'''_T(k-k_T)^3, \quad \text{if} \quad k > k_I,$$

and

$$\text{(ii)}\quad \phi(k) \doteqdot \phi_D + \tfrac{1}{2}\phi''_D(k-k_D)^2 + \tfrac{1}{6}\phi'''_D(k-k_D)^3, \quad \text{if} \quad k < k_I.$$

Figure 1 illustrates this. The expression for ζ is written $\zeta = \zeta_T + \zeta_D$, where

$$\zeta_T = \int_{k_I}^{\infty} \quad \text{and} \quad \zeta_D = \int_{-\infty}^{k_I}.$$

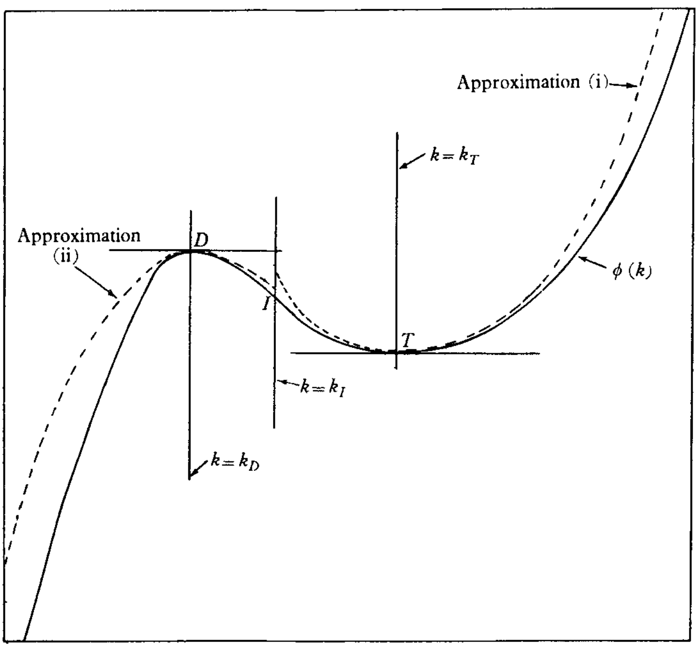

FIGURE 1. An illustration of the approximation to $\phi(k)$ when $|\theta| < \theta_c$. The approximation is most accurate where the rate of change of phase is zero.

It is assumed that $\psi(k)$ is sensibly constant in the region of stationary phase, and we have for large values of x

$$\begin{aligned}\zeta_T &= \mathrm{Im}\int_{k_I}^{\infty} \psi(k)\, e^{ix\phi(k)}\, dk \\ &= \mathrm{Im}\, \psi_T\, e^{i\phi_T x}\int_{k_I - k_T}^{\infty} \exp\{ix(\tfrac{1}{2}\phi''_T k^2 + \tfrac{1}{6}\phi'''_T k^3)\}\, dk.\end{aligned}$$

Following Scorer (1950), we now replace the lower limit of integration by $-\phi''_T/\phi'''_T$. This does not effect the result because the contribution from the interval $(k_I - k_T, -\phi''_T/\phi'''_T)$ is negligible when x is large. We may now write ζ in the form

$$\zeta_S \sim \operatorname{Im} \pi\psi_S \left(\frac{2}{\phi'''_S x}\right)^{\frac{1}{3}} \exp\left[ix\left\{\phi_S + \frac{1}{3}\frac{(\phi''_S)^3}{(\phi'''_S)^2}\right\}\right]\{\operatorname{Ai}(-q_S) \pm i\operatorname{Gi}(-q_S)\},$$

where $q_S = (\phi''_S)^2 [x/2(\phi'''_S)^2]^{\frac{2}{3}}$. Here and after where an alternative sign is attached to a symbol the upper and lower sign correspond to $S = T, D$ respectively. This approximation also holds when T, I and D coincide for when $|\theta| = \theta_c$. The expression is simplified if we write

$$\psi(k) = A(k)\, e^{ia(k)},$$

and

$$\operatorname{Ai}(z) + i\operatorname{Gi}(z) = K(z)\, e^{i\kappa(z)}.$$

Scorer has tabulated $K(z)$ and $\kappa(z)$ for $|z| \leqslant 6$. We may then write

$$\zeta_S \sim A_S \left(\frac{2}{\phi'''_S x}\right)^{\frac{1}{3}} K(-q_S) \sin\{\phi_S x \pm \tfrac{2}{3} q_S^{\frac{3}{2}} \pm \kappa(-q_S) + a_S\}. \tag{1}$$

As usual, ζ_T and ζ_D are called the transverse and divergent wave systems respectively. If the stationary phase approximations to $K(-q_S)$ and $\kappa(-q_S)$ are substituted in this formula, the usual stationary-phase approximations to ζ_T and ζ_D are obtained. For $K(-q_S) \sim (\pi^2 q_S)^{-\frac{1}{4}}$ and $\kappa(-q_S) \sim -\frac{2}{3} q_S^{\frac{3}{2}} + \frac{1}{4}\pi$ as $q_S \to \infty$. By (1) we then have

$$\zeta_S \sim A_S \left(\frac{2\pi}{x|\phi''_S|}\right)^{\frac{1}{2}} \sin(\phi_S x \pm \tfrac{1}{4}\pi + a_S)$$

for large values of q_S.

The method may be justified briefly as follows. One of two conditions is to be satisfied. The first is that the ϕ''_S should be sufficiently small, so that both

$$\exp\left[ix\left\{\frac{\phi^{\text{iv}}_S}{4!}\left(-\frac{\phi''_S}{\phi'''_S}\right)^4 + \frac{\phi^{\text{v}}_S}{5!}\left(-\frac{\phi''_S}{\phi'''_S}\right)^5 + \dots\right\}\right] \doteqdot 1, \tag{2}$$

and

$$\left|(k_I - k_S) + \frac{\phi''_S}{\phi'''_S}\right| \leqslant \{|\phi''_S x|^{-\frac{1}{2}}, |\phi'''_S x|^{-\frac{1}{3}}\}_{\min} \tag{3}$$

hold. Here (2) expresses the condition that the cubic approximation to ϕ is sufficiently accurate in the neighbourhood of stationary phase, and (3) expresses the condition that the contribution from the intervals $(k_I - k_S, -\phi''_S/\phi'''_S)$ are negligible. It is easily found that the length of this interval is $O(\phi''_S)^2$ when the ϕ''_S are small, so that condition (3) is weaker than (2). Hence the first condition may be written

$$(\phi''_S)^4 x \ll 1. \tag{4}$$

The alternative, which is the condition for the validity of the usual method of stationary phase, is that the ϕ''_S should be sufficiently large so that

$$|\phi''_S|^3 x \gg 1. \tag{5}$$

The regions in which the alternatives (4) and (5) hold overlap if x is sufficiently large, and the approximation (1) is then valid for all $|\theta| \leqslant \theta_c$.

For values of $|\theta| > \theta_c$, $\phi(k)$ has no turning point and we write

$$\phi(k) \sim \phi_I + \phi'_I(k - k_I) + \tfrac{1}{6}\phi'''_I(k - k_I)^3,$$

for all values of k. Figure 2 shows this (cf. Hogner 1923, equation (33)). This approximation may be regarded as a further extension of the principle of stationary phase. For although here no turning point exists, the rate of change of phase is sufficiently small if θ is sufficiently close to θ_c. We have

$$\begin{aligned}\zeta_T &= \operatorname{Im}\int_{k_I}^{\infty}\psi(k)\,e^{ix\phi(k)}\,dk\\ &\sim \operatorname{Im}\psi_I\,e^{ix\phi_I}\int_0^{\infty}\exp\{i(\phi_I' k+\tfrac{1}{6}\phi_I''' k^3)\}\,dk\\ &= \operatorname{Im}\pi\psi_I\left(\frac{2}{\phi_I''' x}\right)^{\frac{1}{3}} e^{i\phi_I x}\{\operatorname{Ai}(p)+i\operatorname{Gi}(p)\},\end{aligned}$$

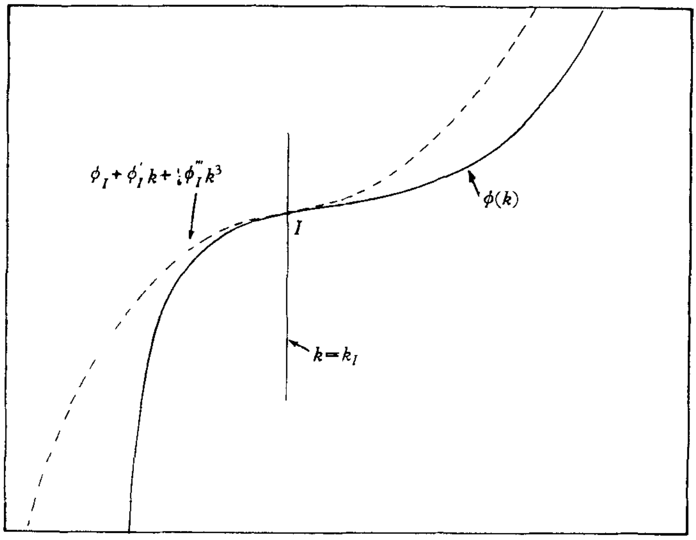

FIGURE 2. The approximation to $\phi(k)$ when $|\theta| > \theta_c$. The approximation is most accurate where the rate of change of phase is least.

where $p = \phi_I'(2x^2/\phi_I''')^{\frac{1}{3}}$. A similar expression for ζ_D holds except that the term in Gi(p) is prefixed by a minus sign. These results for ζ_T and ζ_D may be added to eliminate the term in Gi(p), and we obtain

$$\zeta = 2\pi A_I\left(\frac{2}{\phi_I''' x}\right)^{\frac{1}{3}}\operatorname{Ai}(p)\sin\{\phi_I x+a_I\}.$$

This expression rapidly tends to zero as the angle θ increases. However, it is of interest to extend the two wave systems beyond the critical line, so that the manner in which the transverse and divergent wave system interfere with each other may be seen. The expressions for ζ_T and ζ_D when $|\theta| > \theta_c$ may be written

$$\zeta_S \doteqdot \pi A_I\left(\frac{2}{\phi_I''' x}\right)^{\frac{1}{3}} K(p)\sin\{\phi_I x\pm\kappa(p)+a_I\}. \tag{6}$$

3. The wave pattern

The surface displacement given by (1) and (6) is calculated in the vicinity of $\theta = \theta_c$ with $x \doteqdot 10^4$. It is assumed that $\psi(k)$ in the region is a constant, which for convenience we take to be π^{-1}. The parameter used for the calculations is Δ, where $\Delta^2 = 1-8\tan^2\theta$. This gives

$$\phi_S = \tfrac{3}{4}(\tfrac{3}{2})^{\frac{1}{2}}(1 \pm \tfrac{1}{3}\Delta)\left(\frac{1 \pm \frac{1}{3}\Delta}{1 \pm \Delta}\right)^{\frac{1}{2}}, \quad \phi''_S = \pm\frac{3\Delta}{2^{\frac{1}{2}}}\left(\frac{1 \mp \Delta}{1 \pm \Delta}\right)^{\frac{1}{2}},$$

$$\phi_I = \tfrac{3}{4}(\tfrac{3}{2})^{\frac{1}{2}}\left(\frac{1-\frac{2}{3}\Delta^2}{1-\Delta^2}\right)^{\frac{1}{2}}, \quad \phi'_I = -\frac{1}{2^{\frac{1}{2}}2}\frac{\Delta^2}{(1-\Delta^2)^{\frac{1}{2}}},$$

$$q_S x^{-\frac{2}{3}} = \frac{\Delta^2}{2\,.\,3^{\frac{1}{3}}}\frac{1 \pm \frac{1}{3}\Delta}{1 \mp \Delta}\left(\frac{1 \mp \Delta}{1 \pm \Delta}\right)^{\frac{1}{2}}, \quad p x^{-\frac{2}{3}} = \frac{-\Delta^2}{2\,.\,3^{\frac{1}{3}}\{(1-\frac{2}{3}\Delta^2)(1-\Delta^2)\}^{\frac{1}{3}}}.$$

The substitution $y = (x/2\sqrt{2})-\eta$ is also made, so that for small Δ,

$$\eta/x = \Delta^2(1+\tfrac{1}{4}\Delta^2)/4\sqrt{2},$$

and for small η, p and q_S are proportional to $x^{-\frac{1}{3}}\eta$. It is interesting that although there is an optimum choice of axes by which the approximation is most accurate, a change of origin shows that the critical line is fixed only in direction. This is illustrated by the extent of the region where the surface displacement is $O(x^{-\frac{1}{3}})$. Ursell (1960, pp. 429, 430) has shown that this region extends to a distance $O(x^{\frac{1}{3}})$ on both sides of the critical line. Here it is shown by the form of p and q. For example, outside the critical line, for values of p less than about 1 (or for values of η in the range $0 > \eta > -\frac{1}{2}x^{\frac{1}{3}}$), ζ_T, ζ_D and ζ are comparable with ζ_c. For small values of Δ a series expansion is made; and for points just within the critical angle, we have

$$\zeta_S \simeq x^{-\frac{1}{3}}\left(1 \pm \Delta + \frac{\Delta^2}{18}\right) K\left[\frac{\Delta^2 x^{\frac{2}{3}}}{2\,.\,3^{\frac{1}{3}}}(1 \pm \tfrac{2}{3}\Delta + \tfrac{2}{3}\Delta^2)\right]$$
$$\times \sin\left\{(\tfrac{3}{2})^{\frac{3}{2}}\left(1+\frac{\Delta^2}{6}+\frac{61}{216}\Delta^4\right)x \pm \kappa\left[\frac{\Delta^2 x^{\frac{2}{3}}}{2\,.\,3^{\frac{1}{3}}}(1 \pm \tfrac{2}{3}\Delta + \tfrac{2}{3}\Delta^2)\right]\right\}.$$

The variation of amplitude and phase with η for the divergent and transverse wave systems is shown in figure 3. ζ_T, ζ_D and ζ may be written in the forms

$$\zeta_S \propto B_S \sin\{\tfrac{1}{2}(\phi_T+\phi_D)x \pm \epsilon_S\},$$

and

$$\zeta \propto B \sin\{\tfrac{1}{2}(\phi_T+\phi_D)x + \epsilon\},$$

so that ϵ is the phase of the combined wave system with respect to the curve $\phi_T+\phi_D =$ const. B has a maximum of about 1·09 at $\eta \doteqdot 13$, a minimum of about 0·06 at $\eta \doteqdot 28$, and at $\eta = 0$, $B \doteqdot 0{\cdot}71$. Since the amplitude B_S and the phases ϵ_S of the two wave systems are comparable with each other in the vicinity of θ_c, the phase of the resultant wave system changes rapidly in regions where ζ_T and ζ_D are opposed, that is, in regions where $\epsilon_T+\epsilon_D = \pi$. Ursell (1960, figures 2–4 and p. 431) shows this and explains this in terms of the zeros of the Airy integral. Here in figure 4 it is shown by a bend in the curve of zero surface displacement in the region of $\eta = 28$. Figure 4 also shows the two wave systems extended beyond the critical line. Here the amplitudes of both waves steadily decrease and the crests of one system rapidly approach the troughs of the other.

The general character of the results is similar to those of Hogner (1923) and Ursell (1960). However, differences in detail are to be expected since a different form of ϕ is used here. There is a corresponding difference in ψ, but this is not taken into account since the nature of ψ depends on the ship, as does the nature of the waves. This partly accounts for the difference between Hogner's curves of

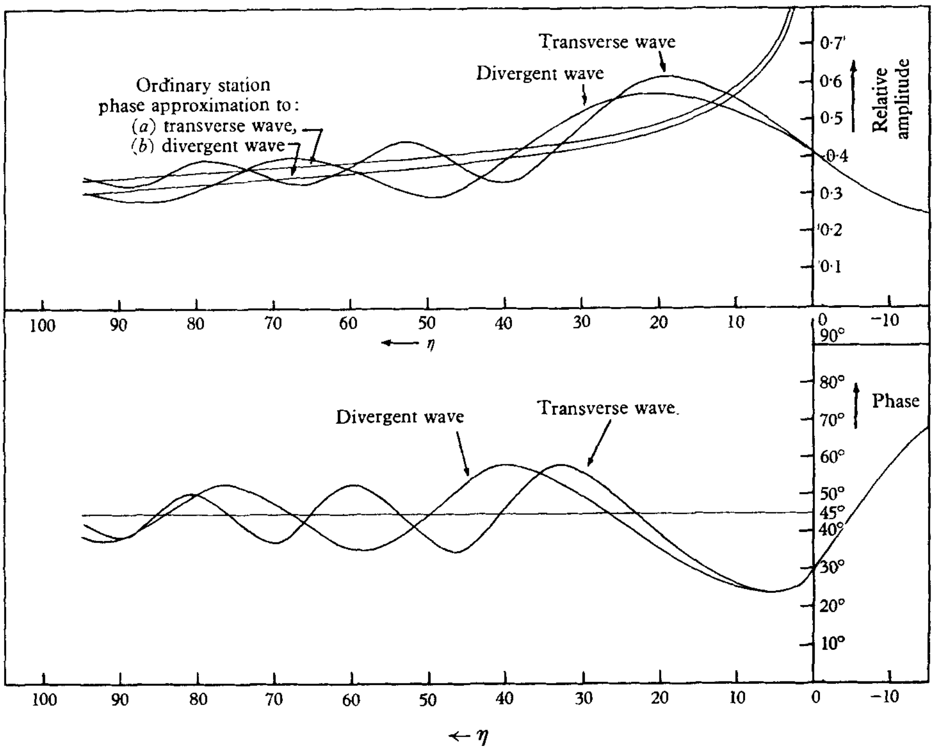

FIGURE 3. Curves of amplitude and phase for the divergent and transverse wave systems in the critical region for $x = 10^4$. The amplitude of both systems are comparable with each other, but the variation of amplitude and phase is more rapid for the transverse waves.

the divergent and transverse wave amplitudes and those shown in figure 3. There is also another reason for this difference. Hogner defines ζ_1 and ζ_2 differently from their counterparts in this paper. In effect the values ζ_T and ζ_D are augmented and diminished respectively by an integral taken along a path in the complex k-plane from the point k_I on the real axis to $-i\infty$. The contours of the system, not shown here, are similar to those given by Ursell (1960, figure 4). Ursell derives for ζ a full asymptotic development whose form remains unchanged across the critical line. However, the simplicity and directness of the method described here have obvious advantages if a first approximation is required.

Further, when $\psi(k)$ is variable the method quickly gives an idea of the magnitude of the transverse and divergent waves over the whole wave pattern. Thus, for ships of broad beam moving at slow speeds, the transverse waves dominate the pattern, but for slender high-speed craft the divergent waves are most prominent. For example,

$$\psi = \exp\left[-\frac{ag}{U^2}\cosh k - \frac{bg}{U^2}\cosh k \sinh |k|\right] \tag{7}$$

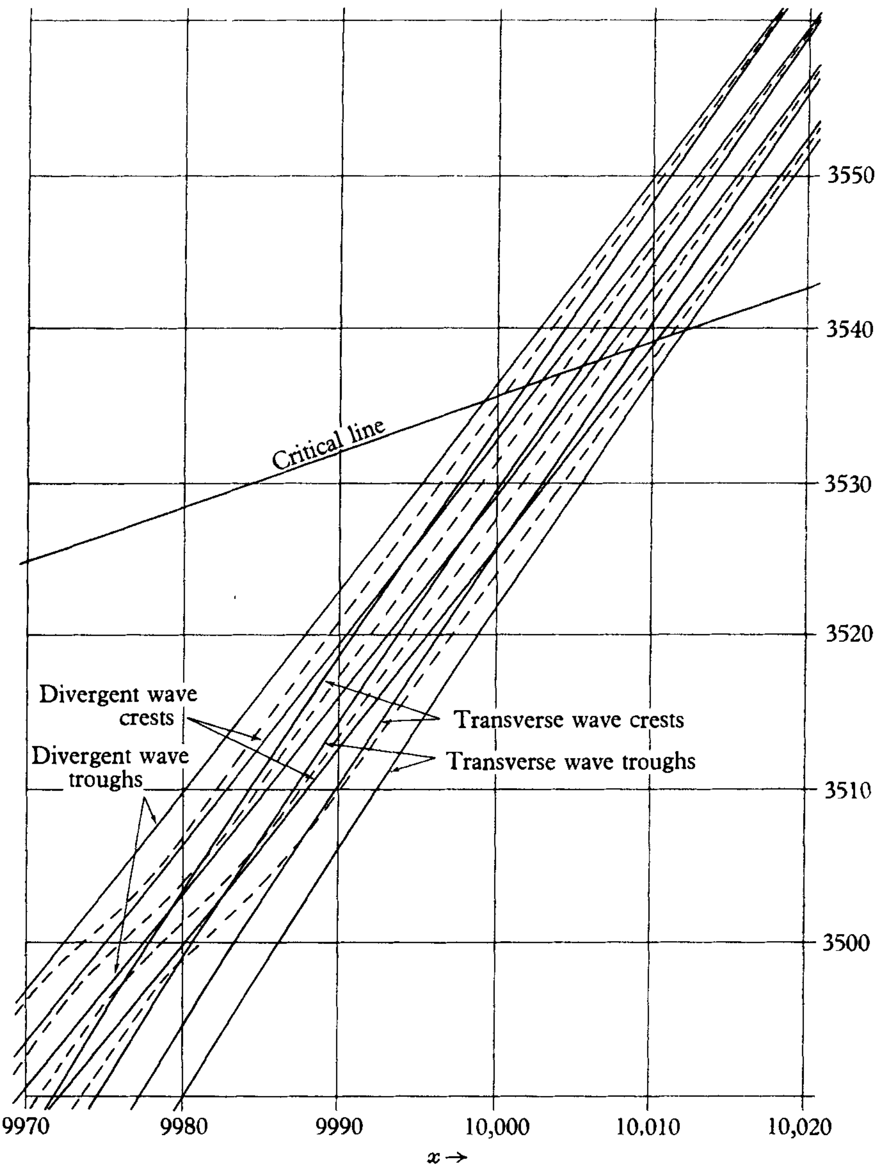

FIGURE 4. Crests and troughs of the two wave systems in a critical region in the vicinity of $x = 10^4$. Curve of zero surface displacement are shown as broken lines. These curves turn sharply in the region of $\eta = 28$, where the two wave systems are opposed, because the two systems have nearly equal amplitudes. Beyond the critical line the amplitudes of both systems are equal and decrease like η^{-1}. Here the troughs of one system rapidly approach the crests of the other and the resultant surface displacement dies away extremely rapidly, like $(\eta x)^{-\frac{1}{4}} \exp(-c\eta^{\frac{3}{2}}x^{-\frac{1}{2}})$, where c is a constant; but for values of η in the range $0 > \eta > -\frac{1}{2}x^{\frac{1}{3}}$, ζ_T, ζ_D and ζ are comparable with ζ_c ($= \zeta$ for $\theta = \theta_c$).

represents a 'forcive' whose strength at any point is proportional to

$$(a^2+x^2)^{-1}(b^2+y^2)^{-1}.$$

This forcive is of infinite extent, but beyond certain points its effect is negligible, and expression (7) therefore represents a disturbance whose beam–length ratio is b/a. If the ship is moving at an angle of yaw β, (7) becomes

$$\psi(k) = \exp\left[-\frac{ag}{U^2}\cosh k\,|\cos\beta-\sin\beta\sinh k|\right] \times \exp\left[-\frac{bg}{U^2}\cosh k\,|\sin\beta+\cos\beta\sinh k|\right].$$

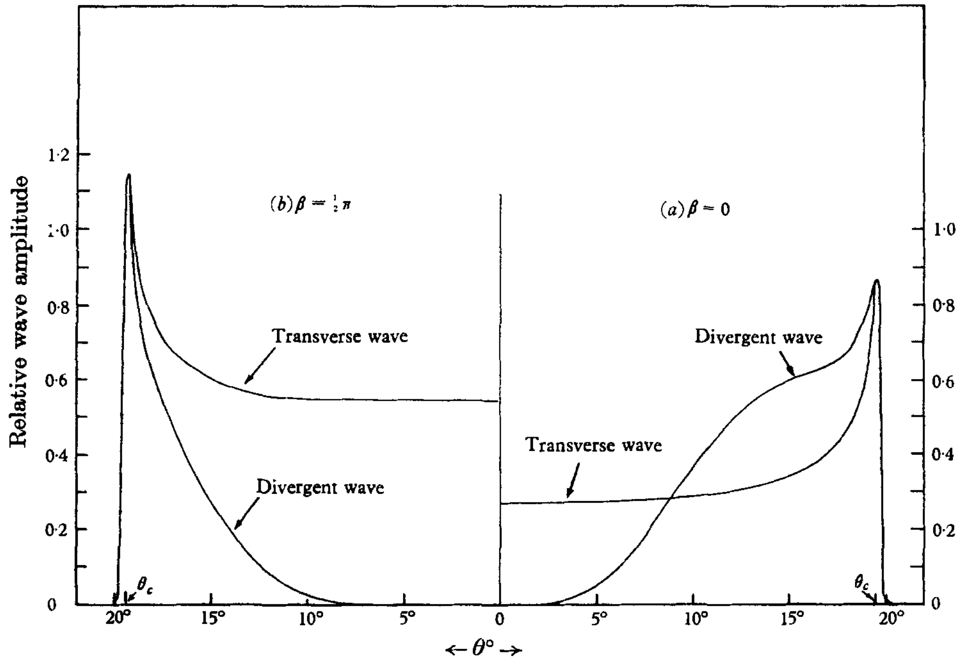

FIGURE 5. Variation of the amplitude of the transverse and divergent waves for a slender high-speed craft, whose beam–length ratio is 1:10. (a) For no yaw, $\beta = 0$. (b) Lateral motion, $\beta = \frac{1}{2}\pi$. This also illustrates the nature of the relative wave amplitudes for a slow moving broad-beamed ship.

In particular for points at θ ($\theta > 0$) well within the critical angle, this gives expressions for the amplitudes of the transverse and divergent waves which are proportional to

$$\exp\left[-a'\left(\frac{1\pm\frac{1}{3}\Delta}{1\pm\Delta}\right)^{\frac{1}{2}}\left|\cos\beta+\frac{1}{2^{\frac{1}{2}}}\sin\beta\left(\frac{1\mp\Delta}{1\pm\Delta}\right)^{\frac{1}{2}}\right|\right]\exp\left[-b'\left(\frac{1\pm\frac{1}{3}\Delta}{1\pm\Delta}\right)^{\frac{1}{2}}\left|\sin\beta-\frac{1}{2^{\frac{1}{2}}}\cos\beta\left(\frac{1\mp\Delta}{1\pm\Delta}\right)^{\frac{1}{2}}\right|\right]\left(\frac{1\pm\frac{1}{3}\Delta}{1\pm\Delta}\right)^{\frac{1}{4}}(\Delta x)^{-\frac{1}{2}},$$

where
$$a' = \frac{ag}{U^2}\left(\frac{3}{2}\right)^{\frac{1}{2}}, \quad b' = \frac{bg}{U^2}\left(\frac{3}{2}\right)^{\frac{1}{2}}.$$

For corresponding points at $-\theta$, we obtain the same expressions for the wave amplitudes except that β is replaced by $-\beta$. The relative amplitudes for $\beta = 0$,

$\beta = \frac{1}{2}\pi$ are shown in figure 5, where values of $a' = 1$ and $b' = 0{\cdot}1$ have been taken. It shows that for such a craft moving ahead at high speed, the divergent waves are most prominent, whereas a broadside current produces mostly transverse waves.

An interesting case arises when $\beta = \tan^{-1}\sqrt{2} \doteqdot 54\frac{3}{4}°$, i.e. when the ship lies parallel to the wave crests at the critical line, for then the largest wave amplitudes occur. For negative θ, very large divergent waves occur which dominate the pattern. When $|\theta|$ is smaller or when θ is positive, however, transverse waves of

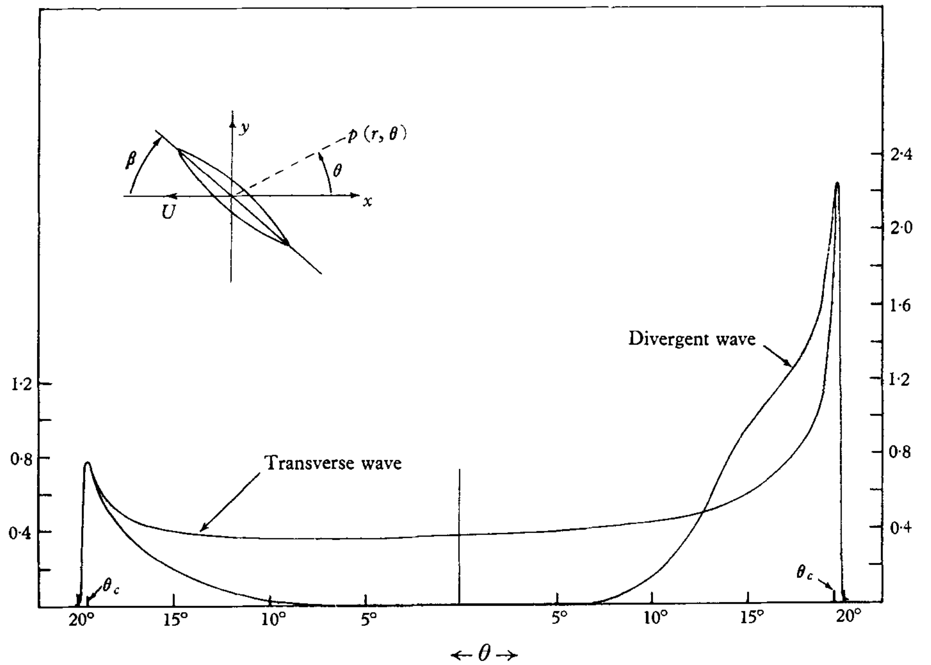

FIGURE 6. Variation of the amplitude of the transverse and divergent waves for a slender high-speed craft whose beam–length ratio is 1:10, for an angle of yaw $\beta = \tan^{-1}\sqrt{2} \doteqdot 54\frac{3}{4}°$ and $x = 10^4$. Here the centre-line of the ship is parallel to the wave crests at the critical line on the after port side.

smaller amplitude are most apparent. Figure 6 illustrates this. These results are similar to those of Scorer (1956*a*, *b*), where lee waves caused by direct and oblique airflow over an isolated hill are discussed.

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